• Graph

1. Graph Isomorphism 图同构　　1, 1, 2, 4, 11, 34…

2. Euler Theorem　① 欧拉回路　节点度数为偶数 & 连通

② 欧拉路径　0/2 个奇度数节点 & 连通　[两奇节点路径 + 欧拉回路]

3. matching　a set $M \subseteq E$ s.t. no edges share an endpoint

① perfect matching　点覆盖

② maximum matching　边最多

③ biparitite matching　[2-colorable / 无奇环]

$\begin{cases} K_n & n\text{ 点完全图} \\ K_{m,n} & \text{二分图 } |V_1|=m,\ |V_2|=n \end{cases}$

Hall Theorem: bipartite graph $G=(V,W,E)$ has a perfect matching iff $|N(S)| \geq |S|$ for each subset S of V and W. [$N(S)=\{v \mid v$ is a neigber of some vertex in $S\}$]

* if $S \subseteq X$　$|N(S)|=|S|$, then $S \leftrightarrows N(S)$ 相互匹配.

if $\exists V_0 \in N(S) \mid$　V_0 matches with $U_0 \in X \setminus S$, then $|S|$ need to match with $|N(S)|-1$ vertex, which is an impossible case (Actually, S only equals $\varnothing$ or X)

④ stable matching

[unstable: $(b,g) \notin M$ but both prefer each other over their current partners in M

Gale-Shapley Theorem: 1° free boys propose to best no-yet-proposed girl

2° each girl keeps her favorite proposal (tentative) and reject others

3° rejected boys try next turn

4° within n^2 turns to be a stable match

4. coloring ① $\chi(G) \geq \omega(G)$　$\omega(G)$ 为最大完全子图

② $\chi(G) \leq k+1$　maximum $\deg(V)=k$

③ interval graph　$\chi(G) \geq \omega(G)$　$\chi(G) \leq \omega(G)$ 按左端点排序. 当染某一节点时, 其相邻节点中左端点小于该点的 $\leq \omega(G)-1$ 自身 $\Rightarrow \chi(G)=\omega(G)$

④ Euler's formula　$V-E+F=2$　k-connected　$V-E+F=k+1$

⑤ planar graph　1° $\sum_{f\in F} \deg(f) = 2E$ 平面图即可

2° $\sum \deg(f) \geq 3F$　简单平面图

推论:

1° $2E \geq 3F$, $V-E+F=2 \Rightarrow E \leq 3V-6$　(简单平面图)

2° 存在度数 ≤ 5 的点 V_0　[$2E=\sum \deg(v) \geq 6V \Rightarrow E \geq 3V$ contradicts with $E \leq 3V-6$]　$\boxed{K_5}$

3° 无三角形存在时　$2E=\sum_f \deg(f) \geq 4F \Rightarrow F \leq \frac{E}{2} \Rightarrow E \leq 2V-4$　$\boxed{K_{3,3}}$

4° 6-coloring　由 2° 可知将 V_0 删除后染色, V_0 必能染色, 重复即可.

• gcd & fibonacci

1. $\gcd(a,b) \mid \gcd(a,bc)$

$\begin{cases} \gcd(a,b) \mid b \Rightarrow \gcd(a,b) \mid bc \\ \gcd(a,b) \mid a \end{cases} \Rightarrow \gcd(a,b) \mid \gcd(a,bc)$

2. $b \mid a \Leftrightarrow \gcd(a,b) = b$

① if $a = kb$ $\gcd(a,b) = \gcd(b, a\%b) = b$

② Assume $b \nmid a$. let $a = kb + r$ $(0 < r < b)$

$\gcd(a,b) = \gcd(b,r) < b$ contradiction! $\Rightarrow b \mid a$

3. if $\gcd(a,c) = 1$, then $\gcd(a,bc) = \gcd(a,b)$

① by 1 we know $\gcd(a,b) \le \gcd(a,bc)$

② $\gcd(ab,bc) = \gcd(a,c) \times b = b \ge \gcd(a,b)$

$\Rightarrow \gcd(a,bc) = \gcd(a,b)$

4. if $b \mid c$, then $\gcd(a,b) = \gcd(a+c,b)$

$\gcd(a+c,b) = \gcd(a\%b, b) = \gcd(a,b)$

5. $\gcd(F_n, F_{n-1}) = 1$ [F_n fibonacci sequence: $\overset{F_0}{0}, 1, 1, 2, 3$]

$\gcd(f_n, f_{n-1}) = \gcd(f_{n-1}, f_n - f_{n-1}) = \gcd(f_{n-1}, f_{n-2}) = \cdots = \gcd(f_1, f_2) = 1$

6. $f_{n+m} = f_{n-1} f_m + f_n f_{m+1}$

method 1: prove by induction

① $m=1$, $f_{n+1} = f_{n-1} \cdot 1 + f_n \cdot 1$

② $m=2$, $f_{n+2} = f_{n-1} + f_3 f_n = f_n + (f_n + f_{n-1}) = f_{n+1} + f_n$

③ $P(m), P(m+1) \Rightarrow P(m+2)$

$f_{n+m+2} = f_{n+m+1} + f_{n+m} = f_{n-1}(f_m + f_{m+1}) + f_n(f_{m+2} + f_{m+1}) = f_{n-1} \cdot f_{m+2} \cdot f_n \cdot f_{m+3}$

method 2:

if $m \mid n$, then $f_m \mid f_n$

$$\begin{bmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n+1}$$

$$f_{n+m} = \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n+m-1} \right)_{1,1} = \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{m-1} \right)_{1,1} = \left(\begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} \begin{bmatrix} f_m & f_{m-1} \\ f_{m-1} & f_{m-2} \end{bmatrix} \right)_{1,1}$$

$$= f_{n-1} f_m + f_n f_{m+1}$$

7. $\gcd(f_n, f_m) = f_{\gcd(n,m)}$ let $m < n$

$\gcd(f_n, f_{n+m}) = \gcd(f_n, f_{n-1} f_m + f_n f_{m+1})$

$= \gcd(f_n, f_{n-1} f_m)$ [since $\gcd(f_n, f_{n-1}) = 1$]

$= \gcd(f_n, f_m)$

$\Rightarrow \gcd(f_n, f_{n+m}) = \gcd(f_n, f_m) = \cdots = \gcd(f_{\gcd(n,m)}, f_0) = f_{\gcd(n,m)}$

8. $\gcd(a^k, b^k) = \gcd^k(a,b)$

$$\gcd(a^k, b^k) = \gcd^k(a,b)\left[\gcd\left(\left[\frac{a}{\gcd(a,b)}\right]^k, \left[\frac{b}{\gcd(a,b)}\right]^k\right)\right] = \gcd^k(a,b)$$